A Sequence of the Extreme Vertices of a Moving Regular Polyhedron Using Spherical Voronoi Diagrams

Hyoung Seok Kim[†]

ABSTRACT

We present an efficient algorithm for finding the sequence of extreme vertices of a moving regular convex polyhedron P with respect to a fixed plane H. The algorithm utilizes the spherical Voronoi diagram that results from the outward unit normal vectors nF_i 's of faces of P. It is well-known that the Voronoi diagram of n sites in the plane can be computed in $O(n\log n)$ time, and this bound is optimal. However, exploiting the convexity of P, we are able to construct the spherical Voronoi diagram of nF_i 's in $O(n)$ time. Using the spherical Voronoi diagram, we show that an extreme vertex problem can be transformed to a spherical point location problem. The extreme vertex problem can be solved in $O(\log n)$ time after $O(n)$ time and space preprocessing. Moreover, the sequence of extreme vertices of a moving regular convex polyhedron with respect to H can be found in $O(\log n + \sum_{j=0}^{s} m_k^j)$ time, where m_k^j $(1 \le j \le s)$ is the number of edges of a spherical Voronoi region $sreg(nF_k^j)$ such that F_k^j gives one or more extreme vertices.

구면 보로노이 다이아그램을 이용한 움직이는 정규 다면체의 근점 알고리즘

김 형 석[†]

요 약

본 논문에서는 고정된 평면에 대한 움직이는 정규 다면체의 가까운 점들을 효과적으로 찾는 알고리즘을 제안한다. 본 알고리즘은 문제를 효율적으로 해결 위하여 다면체의 각 면에서 정의되는 단위 법선 벡터들에 의해 구성되는 구면 보로노이 다이아그램을 이용한다. 일반적인 보로노이 다이아그램이 $O(n\log n)$ 시간에 구성되는 것에 반하여 여기에 사용되는 구면 보로노이 다이아그램은 $O(n)$에 구할 수 있음을 보인다. 이를 본 문제에 적용하면 구면 위치 파악 문제로 전환할 수 있다. 따라서, 주어진 시점에서의 근점은 $O(\log n)$ 시간에 구할 수 있고, 다면체의 움직임에 따라 변하는 근점들의 리스트는 $O(\log n + \sum_{j=0}^{s} m_k^j)$ 시간에 구할 수 있다. 이때, m_k^j $(1 \le j \le s)$는 질의점이 지나는 구면 보로노이 다이아그램의 영역 $sreg(nF_k^j)$의 선분의 개수이다. 본 논문에서 제안한 알고리즘은 컴퓨터 애니메이션과 로보틱스 분야에서 충돌지점을 찾는 문제에 효과적으로 사용될 수 있다.

1. Problem Definition

The minimum Euclidean distance problem from a 3D convex object P to an infinite plane H can be reduced to an extreme vertex problem, i. e., identifying a vertex of P that first touches H when H is translated toward P. Edelsbrunner [1] was able to solve this problem in $O(\log n)$ time after $O(n)$ time and space preprocessing, where n is the number of vertices in P. The efficiency of his method is due to an elegant data structure representing a hierarchy of convex polyhedra, $(P_0, P_1, \cdots, P_l)$ nested in P such that $P_0 = P$, P_i

[†] 정회원, 동의대학교 컴퓨터응용공학부 전임강사

is a tetrahedron, $l = O(\log n)$, and $P_{i+1} \subset P_i$ for all $0 \le i < l$. In order to find the extreme point of P with respect to H, the extreme vertex a_l of the innermost polyhedron P_l is first identified in $O(1)$ time. a_l gives a constant number of candidate vertices of P_{l-1} to check for extremity. This yields a_{l-1} from which a_{l-2} can be similarly derived, and so on. It takes a constant time to move from one polyhedron to the next. Since $l = O(\log n)$, the extreme vertex a_0 of P with respect to H can be found in $O(\log n)$ time.

In this paper, we are concerned with a more generalized version of the extreme vertex problem with the object in motion. Rigid motion consists of translation and rotation. Since H is a fixed infinite plane, the extreme vertex of the object P with respect to H is translation-invariant. Therefore, it is sufficient to consider only rotation for our purpose.

Let $Q(t)$, $0 \le t \le 1$ be a curve that gives the orientation of P at time t [2]. The single rotation about a unit vector u by angle θ can be compactly given by a vector $r = \theta u$, where $u = r/\|r\|$ and $\theta = \|r\|$. Let $0 \le t_i \le 1$, $i = 0, 1, \cdots, m$, be a sequence of discrete times such that $0 = t_0 < t_1 < \cdots < t_m = 1$. By sampling $Q(t_i)$ at carefully chosen t_i's, $Q(t)$ can be approximated by a sequence of rotation vectors, $(r_0, r_1, \cdots, r_{m-1})$, where the $i+1^{th}$ rotation vector r_i gives a single rotation about r_i by $\|r_i\|$ radians during the time intervals $t_i \le t \le t_{i+1}$[3].

Denoting by $R_{i,j}$ an orthonormal matrix representing the sequence of rotations by $(r_i, r_{i+1}, \cdots, r_j)$, $R_{0,i} = R_{i,i} \cdot R_{0,i-1}$, where $R_{i,i}$ is a single rotation given by r_i. Now, our extreme vertex problem for a moving object can be abstracted as follows: Given a fixed infinite plane H, an object P with a composite rotation matrix R, and a rotation vector r, find the sequence of extreme vertices of P with respect to H while P, whose initial orientation is given by R, is rotated about r by $\|r\|$ in a time interval [0, 1] with a constant speed.

In this framework, Edelsbrunner's original problem can be viewed as a special case of our problem when r is a zero vector meaning no rotation at all. Although the nested polyhedral hierarchy gives an optimal solution for the special case, it is not apparent to adapt the hierarchy to our problem. We employ the notion of the Gauss sphere [4] to construct spherical Voronoi diagrams for efficiently solving both the original problem and its generalized version.

The rest of this paper is organized in the following manner : In Section 2, we explain the basic idea of this paper and show that an extreme vertex problem can be transformed to a spherical point location. In Section 3, we also show that the spherical Voronoi diagram of n outward unit normal vectors of faces in P can be computed in $O(n)$ time. Section 4 presents an algorithm for solving the generalized extreme vertex problem using the spherical Voronoi diagram. Finally, we conclude this paper with some remarks in Section 5.

2. Basic Idea

It is well-known that Voronoi diagrams play a central role for solving a variety of proximity problems [5,6]. In this section, we show that our extreme vertex problem can be transformed to a series of two closest point problems on 3D and 2D spheres, respectively. We also show that these proximity problems are essentially reduced to point location problems by employing 3D and 2D spherical Voronoi diagrams, in which the proximity information among the faces and edges on the boundary of the convex hull of a 3D object P is embedded, respectively.

We start with characterizing a face of P, that contains an extreme vertex of P with respect to the plane H. Let the boundary of P consist of n convex faces, F_i, $i = 1, 2, \cdots n$. We denote the outward unit normal vector of F_i by nF_i for each i. Suppose that P is sufficiently apart from H so that P always lies above H, which makes sense

if we regard H as the surface of the ground. The unit normal vector of H pointing below is denoted by nH (see Figure 1). We establish a relationship between nH and the normal vector nF_k of the face F_k of P containing an extreme vertex.

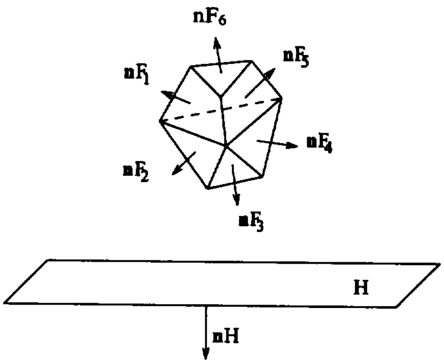


Fig. 1. Configuration of a plane H and a polyhedron P.

Lemma 1 A face F_k of a regular convex polyhedron P contains an extreme vertex of P with respect to H if

$$\langle nH, nF_k \rangle = \max_i \langle nH, nF_i \rangle, \qquad (1)$$

where $\langle nH, nF_i \rangle$ denotes the inner product of two unit vectors, nH and nF_i. Moreover, all extreme vertices of P lies on such a face F_k.

[Proof] Let F_k satisfy the condition given by Equation (1). Suppose, for contradiction, that F_k does not contain any extreme vertices of P with respect to H. Let v_j be the closest point to H that lies on F_k. Since nF_k satisfies Equation (1), a plane H' that is parallel to H, locally supports P at v_j, i. e., all faces containing v_j lie above H'. Let v^* be an extreme vertex of P with respect to H. Since v^* is closer to H than v_j is, v^* lies below H'. Consider the line segment joining v^* and v_j. The portion of the line segment that is contained in an open neighborhood $B(v_j, \varepsilon)$ for arbitrarily small $\varepsilon > 0$ does not belong to P. This contradicts the fact that P is convex. Hence, F_k must contain an extreme vertex. In order to prove the second part of the lemma, consider H'. Since F_k has an extreme point v^* with respect to H, P lies above H'. H' may contain v^*, an edge containing v^*, or F_k itself. In any cases, all extreme vertices lie in F_k. □

Lemma 1 suggests how to find the face F_k containing all extreme vertices of P with respect to H ; the inner product of nH and nF_k is the maximum over all inner products of nH and nF_i for all $i = 1, 2, \cdots, n$. Let θ_i be the angle between nH and nF_i. Then,

$$\cos\theta_i = \langle nH, nF_i \rangle \quad \text{for } 0 \le \theta_i \le \pi.$$

Therefore, Equation 1 can be rewritten as follows:

$$\cos\theta_k = \max_i \cos\theta_i, \text{ and } 0 \le \theta_i \le \pi, \; i = 1, 2, \cdots, n$$

Since $\cos\theta_i$ is monotonically decreasing in θ_i for $0 \le \theta_i \le \pi$, θ_k is the minimum angle between nH and nF_i for all i.

nH and nF_i's are unit vectors, that can be mapped onto points on the Gauss sphere [4], i. e., a unit sphere centered at the origin. Let C_i be the unit circle containing nH and nF_i on the sphere. Then, θ_i can be measured by the length of the shorter arc on C_i that connects two points nH and nF_i on the sphere. Therefore, the problem of finding F_k can be reduced to a closest point problem on the sphere. Now, in order to identify F_k, we determine the closest point nF_k to nH among the points, nF_i, $i = 1, 2, \cdots, n$, on the sphere. This can be transformed to a point location problem using the spherical Voronoi diagram of nF_i's.

Suppose that the face F_k is found. We still need to identify an extreme vertex of P among the vertices in F_k. In the worst case, the number of vertices in F_k is $O(n)$, which causes an extra cost. Clearly, F_k is a convex polygon. Consider the plane H_k containing F_k. Assuming that H_k and H are not parallel, let L_k be the intersection line

of H and H_k (see Fig. 2).

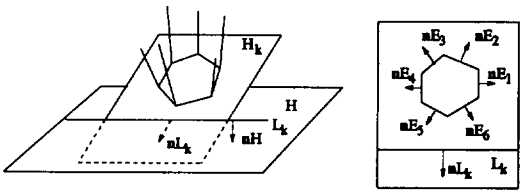


Fig. 2. The intersection line L_k of two planes H and H_k.

If H_k and H are parallel, then L_k is not well-defined. In this case, every vertex of F_k is an extreme vertex of P with respect to H. Obviously, an extreme vertex is closer to L_k than the others in F_k are. The problem of finding an extreme vertex of P with respect to H is reduced to a similar problem of lower dimension, i. e., finding an extreme vertex of the convex polygon F_k with respect to a line L_k.

Let the boundary of F_k be represented by a cycle of vertices and edges $(v_1, E_1, \cdots, v_{n_k-1}, E_{n_k-1}, v_{n_k}, E_{n_k}, v_1)$, where v_i and E_i, $i=1,2,\cdots,n_k$, denote the i^{th} vertex and edge, respectively, and n_k is the number of vertices in F_k. E_i connects two adjacent vertices, v_i and v_{i+1}, where the subscripts are taken modulo n_k. E_i can be considered as a 2D face, and thus its outward unit normal vector nE_i is well-defined. That is, nE_i is the unit vector on H_k, which is perpendicular to E_i and pointing to the outside of F_k. Similarly, nL_k is the unit vector on H_k that is perpendicular to L_k and pointing downward. From Lemma 1, we can directly characterize the edge E_m, $1 \le m \le n_k$, containing an extreme vertex of P.

Corollary 1 Suppose that H_k and H are not parallel. The edge E_m of F_k contains an extreme vertex of P if $\langle nL_k, nE_m \rangle = \max_i \langle nL_k, nE_i \rangle$.

The extreme vertex(vertices) of P can be determined in $O(1)$ time from E_m; either v_m or v_{m+1} (or both). By the similar argument for finding F_k, Corollary 1 reduces the problem of finding E_m to a 2D closest point problem on the unit circle centered at the origin, i. e., the 2D Gauss sphere. That is, given a set of n_k points nE_i, $i=1,\cdots,n_k$ on the unit circle centered at the origin, we find the point closest to nL_k. This problem can be transformed to a point location problem after constructing the 2D spherical Voronoi diagram from those n_k points. Since nE_i, $i=1,\cdots,n_k$, are sorted in accordance with their subscripts along the boundary of F_k, their spherical diagram is obtained in $O(n_k)$ time by simply bisecting every arc on the unit circle, that connects nE_i and nE_{i+1} for $1 \le i \le n_k$. Moreover, the 2D spherical Voronoi diagrams for all faces of P can initially be computed in $O(n)$ time and space as preprocessing.

3. Spherical Voronoi Diagram

Suppose that we are given a set S of n sites on the unit sphere S^2. For two distinct sites $p, q \in S$, the **spherical dominance** of p to q, denoted by $sdom(p,q)$, is defined as the subset of the sphere being at least as close to p as to q. Formally, $sdom(p,q) = \{x \in S^2 \mid d(x,p) \le d(x,q)\}$, where $d(x,y)$ is the Euclidean distance between x and y. Clearly, $sdom(p,q)$ is a closed half sphere divided by the perpendicular bisector of p and q. This bisector separates all points of the sphere closer to p from those closer to q and will be termed the **separator** of p and q. The **spherical region**, $sreg(p) = \bigcap_{q \in S - \{p\}} sdom(p,q)$ of a site $p \in S$, is the portion of the sphere lying in all of the spherical dominances of p over the remaining sites in S. The n spherical regions form a partition of the unit sphere. This partition is called the **spherical Voronoi diagram**, $V(S)$, of the finite point set S.

Brown [5] presented an $O(n \log n)$ algorithm for computing the spherical Voronoi diagram of n points on the unit sphere. His algorithm consists

of three major steps: The first step is to construct the convex hull of n points in $O(n\log n)$ time. The next step is to compute the spherical Voronoi vertices. Let u_i be the point on the sphere that is equi-distant from the three vertices of F_i. Then, u_i is a spherical Voronoi vertex. It takes $O(n)$ time to compute all Voronoi vertices. The final step is to connect the Voronoi vertices in $O(n)$ time: Two Voronoi vertices, u_i and u_j are connected by an arc on a great circle (geodesic arc) if and only if F_i an F_j share an edge. Clearly, the time complexity of the first step dominates those of the others. Hence, if the first step can be done in $O(n)$ time, then we can construct the spherical Voronoi diagram in $O(n)$ time. We exploit the point-plane duality to achieve this time bound.

Consider a transformation that maps a point $p=(p_1, p_2, p_3)$ to a plane $\langle p, x\rangle = p_1x_1 + p_2x_2 +, p_3x_3=1$, and vice versa [7]. This transformation gives a dual D of the polyhedron P : Every vertex v_k of P corresponds to a face Dv_k of D, and every face F_i of P does to a vertex DF_i of D. Without loss of generality, we may assume that P contains the origin in its interior. Otherwise, we can always translate P to satisfy this assumption, since an extreme vertex with respect to H is translation-invariant. It is well known that D is also a convex polyhedron containing the origin in its interior. Moreover, Dv_k is a convex polygon for all $v_k \in P$.

For convenience, we relabel the faces containing $v_k=(x_k, y_k, z_k)$ in P so that they form a cycle $(F_{k,0}, F_{k,1}, \cdots, F_{k,k-1})$, i. e., $F_{k,j}$ and $F_{k,j+1}$ for $0 \le j < k$ share an edge of P, where the second subscripts are taken modulo k . v_k is transformed to a plane

$$\langle v_k, x\rangle = x_kx_1 + y_kx_2 + z_kx_3 = 1.$$

Since $nF_{k,j}$ is the unit normal vector of $F_{k,j}$ of P, the plane containing $F_{k,j}$ is given by

$$\langle nF_{k,j}, x\rangle = d_{k,j}, \quad \text{or w} \langle \frac{nF_{k,j}}{d_{k,j}}, x\rangle = 1.$$

Therefore, $DF_{k,j} = nF_{k,j} \ / \ d_{k,j}$ is a vertex of Dv_k corresponding to $F_{k,j}$ of P. Furthermore,

$$\langle v_k, x\rangle = \|v_k\| \langle \frac{v_k}{\|v_k\|}, x\rangle \ = \ 1, \text{ or}$$

$$\langle \frac{v_k}{\|v_k\|}, x\rangle = \frac{1}{\|v_k\|}.$$

That is, the convex polygon Dv_k corresponding to a vertex v_k of P is contained in the plane $\langle v_k, x\rangle = 1$ whose distance from the origin is $1 \ / \ \|v_k\|$. We can easily verify that $DF_{k,j}$ for $0 \le j < k$ lies on the plane $\langle v_k, x\rangle = 1$:

$$\langle v_k, DF_{k,j}\rangle \ = \ \langle v_k, \frac{nF_{k,j}}{d_{k,j}}\rangle = 1,$$

since $v_k \in F_{k,j}$.

Due to the point-plane duality, $DF_{k,j}$'s are the vertices of the convex polygon Dv_k, and the line segment joining two vertices, $DF_{k,j}$ and $DF_{k,j+1}$, is an edge of Dv_k. As shown in Figure 2, the ray from the origin to $DF_{k,j}$ intersects the Gauss sphere at $nF_{k,j}$. Therefore, Dv_k is projected onto the sphere as a spherical region whose boundary can be represented by a sequence of points $(nF_{k,0}, nF_{k,1}, \cdots, nF_{k,k-1})$.

Let $H_{k,j}$ be the plane containing the origin, $nF_{k,j}$, and $nF_{k,j+1}$, $0 \le j < k$. $H_{k,j}$ divides the space into two half-spaces, $H^+_{k,j}$ and $H^-_{k,j}$. Let $H^+_{k,j}$ be the half-space containing Dv_k. The intersection of half-spaces $H^+_{k,j}$, $0 \le j < k$, is a cone with the apex at the origin. Therefore, the spherical region, that is the projection of Dv_k onto the sphere, is the intersection of the sphere and the cone. The boundary of this region is represented by a sequence of points $(nF_{k,0}, nF_{k,1}, \cdots, nF_{k,k-1})$ such that $nF_{k,j}$ and j $nF_{k,j+1}$ are joined by a geodesic arc for all $0 \le j < k$. Moreover, the spherical region is a simple spherical polygon on the sphere. Hence, the sequence of the line segment joining $nF_{k,j}$ and $nF_{k,j+1}$, $0 \le j < k$, forms a simple closed piecewise linear curve. We show that its orthographic projection onto a plane parallel to Dv_k is a convex polygon:

Lemma 2 The orthographic projection of the

closed piecewise linear curve $(nF_{h,0}, nF_{h,1}, \cdots nF_{h,k-1})$ onto a plane parallel to Dv_h is a convex polygon.

[Proof] Let $(n\overline{F_{h,0}}, n\overline{F_{h,1}}, \cdots, n\overline{F_{h,k-1}})$ be the projection of the curve, where $n\overline{F_{h,j}}$, $0 \leq j < k$, is the projected image of $nF_{h,j}$. Suppose that the projection is not convex. Since it is a simple closed curve, there would exist one or more vertices of the projection contained in the interior of the convex hull of the projection. Take any of such vertices, say $n\overline{F_{h,j}}$ for some $0 \leq j < k$. Then, it must be contained in the interior of the triangle $(n\overline{F_{h,a}}, n\overline{F_{h,b}}, n\overline{F_{h,c}})$, where $0 \leq a < b < c < k$. The inverse projection of the triangle onto the sphere gives a spherical triangle $(nF_{h,a}, nF_{h,b}, nF_{h,c})$, that contains $nF_{h,j}$ as an interior point. When the spherical triangle is transformed back onto the plane containing Dv_i, $DF_{h,j}$ lies in the interior of the triangle $(DF_{h,a}, DF_{h,b}, DF_{h,c})$, that is completely contained in Dv_i. Thus, $DF_{h,j}$ is an interior point of Dv_i, which contradicts that Dv_i is a convex polygon. Hence, the result holds true. □

Aggarwal et. al [8] showed that the convex hull of n points can be found in $O(n)$ time if their projections onto a plane are the vertices of a convex polygon. By Lemma 2, the points $nF_{h,j}$, $0 \leq j < k$, satisfy this condition. Therefore, their convex hull can be constructed in $O(k)$ time. The piecewise linear curve has another nice property that is very useful to construct the convex hull of nF_i's in $O(n)$ time.

Lemma 3 The closed piecewise linear curve $(nF_{h,0}, nF_{h,1}, \cdots, nF_{h,k-1})$ consists of a subset of edges of the convex hull of nF_i for all $i = 1, 2, \cdots, n$.

[Proof] Since nF_i's lie on the Gauss sphere, all of them are extreme points, i. e., the vertices of the convex hull of nF_i's. Since $nF_{h,j} \in \{nF_1, nF_2, \cdots, nF_n\}$ for all $0 \leq j < k$, $nF_{h,j}$'s are also extreme points. We will be done if we show that the line segment joining $nF_{h,j}$ and $nF_{h,j+1}$ is an edge of the convex hull of nF_i's.

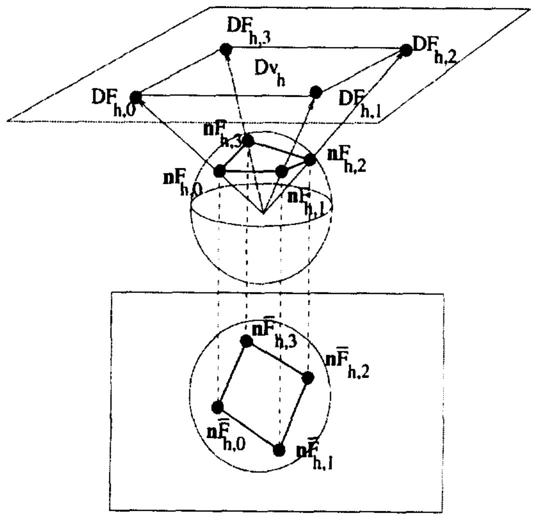


Fig. 3. Point-plane duality and projections.

Remember that the projection of each face Dv_h of the dual D of the convex polyhedron P is a spherical region $(nF_{h,0}, nF_{h,1}, \cdots, nF_{h,k-1})$. This region is the intersection of the Gauss sphere and the cone bounded by the planes $H_{h,j}$ for all $0 \leq j < k$. The geodesic arc connecting $nF_{h,j}$ and $nF_{h,j+1}$ lies on $H_{h,j}$, and so does the line segment joining them. For every $v_h \in P$, the cone CO_h is well- defined, i. e., $CO_h = \bigcap_j H^+_{h,j}$. The set of all these cones partitions the sphere into n disjoint spherical regions. Any of the spherical regions does not contain a spherical image nF_i, $1 \leq i \leq n$ in its interior.

Suppose, for a contradiction, that a line segment of the curve is not an edge of the convex hull of nF_i's, say the line segment joining $nF_{h,l}$ and $nF_{h,l+1}$ for some $0 \leq l < k$. Then, it must be a diagonal. Therefore, the line segment excepting its end points $nF_{h,l}$ and $nF_{h,l+1}$ is completely contained in the interior of the cone CO_h. If we project the line segment back to Dv_h, it becomes a diagonal of Dv_h, which is a contradiction since the inverse projection of the line segment is an edge

of the convex polyhedron Dv_k. Hence, the result follows immediately. □

Now, we are ready to describe how to construct the convex hull of the Gauss images nF_i's of the normal vectors of the faces of P in $O(n)$ time. By Lemma 3, the set of all such curves, that result from the faces of D, partitions the boundary of the convex hull into n disjoint regions. Lemma 2 guarantees that the convex hull of each of these regions can be found in linear time. Therefore, the convex hull of nF_i's can be constructed in $O(n)$ time once all such curves are identified. The curves are obtained in $O(n)$ time by simply projecting the faces of D onto the Gauss sphere with the origin as the projection center. Given the convex hull of nF_i's, the second and third steps of Brown's algorithm take care of the remainder to construct the spherical Voronoi diagram of nF_i's in $O(n)$ time.

Theorem 1 The convex hull of nF_i's, $1 \leq i \leq n$, can be found in $O(n)$ time. Moreover, their spherical Voronoi diagram can be constructed in the same time bound.

4. Algorithm and Analysis

We first solve the extreme problem in the translational case: Find the extreme vertex (vertices) of a regular convex polyhedron P with respect to a fixed infinite plane H when P is translated toward H. This problem can be transformed into two point location problems: one to identify the face containing an extreme vertex and the other to find the edge with the same property.

Given the spherical Voronoi diagram of nF_i, $1 \leq i \leq n$, the former problem can be solved in $O(\log n)$ time [6]. From Theorem 1, the Voronoi diagram can be constructed in $O(n)$ time. Therefore, it takes $O(n)$ time to identify the face F_k containing an extreme vertex. If F_k and H are parallel, then we are done: All the vertices of F_k are extreme. Otherwise, we need to solve the latter problem. From the edges of F_k, the 2D spherical Voronoi diagram can initially be constructed in $O(n_k)$ time as preprocessing. The edge E_m containing an extreme vertex can be found in $O(\log n_k)$ time from this Voronoi diagram in the similar fashion. Finally, it takes constant time to determine the extreme vertex (vertices) from E_m. Hence, the following results hold true:

Theorem 2 The extreme vertex (vertices) of a regular convex polyhedron P with respect to a fixed plane H can be found in $O(\log n)$ time after $O(n)$ time preprocessing when the orientation of P is also fixed.

The Voronoi diagram also plays a central role to solve our generalized problem: Given a fixed infinite plane H, an object P with a composite rotation matrix R, and a rotation vector r, find the sequence of extreme vertices of P with respect to H, while P with its initial orientation given by R is rotated about r by $\|r\|$ in a time interval [0, 1] with a constant speed.

While H is fixed, P rotates about r by $\|r\|$. Every extreme vertex of P with respect to H is dependent on the relative orientation of P to H. Therefore, we can fix P and rotate H about $-r$ by $\|r\|$ for our purpose. Without loss of generality, we assume that H moves around P, hereafter. Under this assumption, nH moves and nF_k's are fixed. The rotation of H for $0 \leq t \leq 1$ gives a curve $H(t)$ representing H at time t and its orientation path $nH(t)$.

Clearly, the initial orientation $nH(0)$ is $R^{-1} \cdot nH$, that is a unit vector corresponding to a point on the Gauss sphere. Since this point rotates about $-r$, $nH(t)$ for $0 \leq t \leq 1$ generates a path on the sphere, that is a circular arc. This path crosses over a sequence of spherical Voronoi regions. While the path is intersecting a spherical Voronoi region $sreg(nF_k)$, a subsequence of extreme ver-

tices are chosen from the vertices of the face F_k. Hence, we need to find the sequence of all time instances at which the path makes a transverse from a region to another.

We first determine the Voronoi region, $sreg(nF_k^j)$, that contains $nH(0)$ and then trace $nH(t)$ on the Gauss sphere to compute the sequence of all time instances. Let $S=(nF_k^0, nF_k^1, \cdots, nF_k^s)$ be the sequence of points such that $nH(t)$ intersects $sreg(nF_k^j)$, $0 \le j \le s$, as time t varies from zero to one. Notice that every pair of adjacent points in the sequence S are distinct. $sreg(nF_k^0)$ can be found in $O(\log n)$ time by solving the point location problem for the query point $nH(0)$ against the spherical Voronoi diagram. We scan every edge of $sreg(nF_k^0)$ to compute the time at which $nH(t)$ first intersects the boundary of $sreg(nF_k^0)$. If $nH(t)$ does not intersect any edge, then it stays on $sreg(nF_k^0)$ for all $0 \le t \le 1$, i. e., $S=(nF_k^0)$. Otherwise, $nH(t)$ leaves $sreg(nF_k^0)$ through the intersection point to enter the next region, $sreg(nF_k^1)$. In general, let $sreg(nF_k^j)$ be the $j+1^{st}$ region intersecting $nH(t)$ during time interval $[t_e^j, t_l^j)$, where t_e^j and t_l^j are the times to enter and to leave $sreg(nF_k^j)$, respectively. Clearly, $t_l^j = t_e^{j+1}$ if $nH(t)$ intersects the boundary of $sreg(nF_k^j)$ after t_e^j. t_l^{j+1} is the time at which $nH(t)$ first intersects the boundary of $sreg(nF_k^{j+1})$ after t_e^{j+1}. For conve nience, we set $t_e^0 = 0$, and $t_l^s = 1$. Moreover, if $nH(t)$ just touches an edge of $sreg(nF_k^j)$ for some $0 \le j \le s$ but does not enter its interior, then $t_e^j = t_l^j$ and thus the internal $[t_e^j, t_l^j]$ degenerates to a time instance. Both $nH(t)$ and every edge of each region are circular arcs. Therefore, their intersection can geometrically be obtained in constant time. Hence, the following result is immediate:

Lemma 4 It takes $O(\log n + \sum_{j=0}^{s} m_k^j)$ time to compute the sequence of all time instances at which $nH(t)$ intersects the boundaries of the spherical Voronoi regions.

Given $S=(nF_k^0, nF_k^1, \cdots, nF_k^s)$ and its corresponding sequence of time intervals $T=([t_e^0, t_l^0), [t_e^1, t_l^1), \cdots, [t_e^s, t_l^s])$, we characterize the solution of the generalized extreme vertex problem, i. e., the sequence of all extreme vertices of P with respect to H. The sequence of all extreme vertices can compactly be represented as an ordered list of subsequences, denoted by $EV=(EV_0, EV_1, \cdots, EV_s)$, where EV_j, $0 \le j \le s$, is the subsequence while $nH(t)$ stays at an $sreg(nF_k^j)$ for $t_e^j \le t \le t_l^j$. Since two or more vertices of P may simultaneously be closest to H at the same time instance, EV_j, $0 \le j \le s$, is represented as a nested ordered list. For example, when $EV_i=((v_a, v_b), v_c, v_d)$, the vertices v_a and v_b of the face F_k^j are extreme, then v_c is, and finally v_d becomes the closest. If $\|r\| \ge 2\pi$, i. e., H rotates about $-r$ more than once, then EV is partially or completely repeated. Without loss of generality, we assume that $\|r\| \le 2\pi$. Otherwise, we can always extend EV by cyclically repeating the vertices in EV. We also assume that P does not hit H. This can easily be checked by examining if the current extreme vertex is below H.

Now, we focus on $sreg(nF_k^j)$ for some $0 \le j \le s$ to characterize EV during $[t_e^j, t_l^j)$. At the time instances t_e^j and t_l^j, we can find the extreme vertices v_0^j and $v_{s_j}^j$, respectively, in $O(\log m_k^j)$ time due to Theorem 2 by solving the original 2D extreme vertex problems since F_k^j is given. Our question is: What are the extreme vertices between v_0^j and $v_{s_j}^j$? One possibility is either the vertices in the boundary of F_k^j from v_0^j and $v_{s_j}^j$ in the counter-clockwise order or those in the clockwise order. The other possibility is : Neither is true.

In order to answer this question, we observe the behavior of $nH(t)$ with respect to nF_k^j. If $nH(t^*) = nF_k^j$ for some $t_e^j \le t^* < t_l^j$, then the face F_k^j is parallel with the plane $H(t)$. Therefore, every vertex of F_k^j is an extreme vertex of P with respect to $H(t)$ at time t^*. t^* can be computed in $O(1)$ time. Suppose that $nH(t) \neq nF_k^j$ for any

$t^j_e \le t < t^j_l$. Then, two planes $H(t)$ and H^j_k containing F^j_k meet along a line $L^j_k(t)$. Denoting by $nL^j_k(t)$ the unit vector that is perpendicular to $L^j_k(t)$ and lies on the plane H^j_k, $nL^j_k(t)$ is orthogonal to nF^j_k. Since three unit vectors nF^j_k, $nH(t)$, and $nL^j_k(t)$ are orthogonal to $L^j_k(t)$, their corresponding points on the Gauss sphere lie on a great circle. The vector $nL^j_k(t)$ is parallel to $nH(t) - \langle nH(t), nF^j_k \rangle nF^j_k$. Since $nL^j_k(t)$ is a unit vector, it is represented by

$$nL^j_k(t) = \frac{nH(t) - \langle nH(t), nF^j_k \rangle nF^j_k}{\| nH(t) - \langle nH(t), nF^j_k \rangle nF^j_k \|} = \frac{nH(t) - \langle nH(t), nF^j_k \rangle nF^j_k}{\sqrt{1 - \langle nH(t), nF^j_k \rangle^2}}. \qquad (2)$$

The vector $nL^j_k(t)$ plays the role of a query point for the 2D spherical Voronoi diagram that is obtained from the unit normal vectors nE_i's of edges on the boundary of F^j_k.

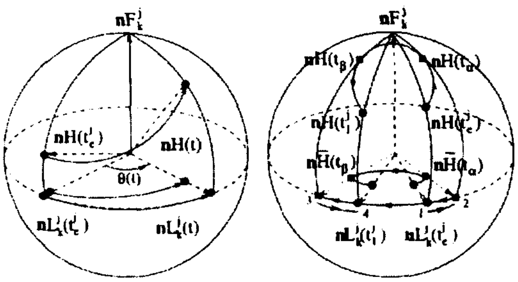
Without loss of generality, let $nF^j_k = (0,0,1)$. Then, $nH(t)$ can be simply represented in the spherical coordinate system:

$$nH(t) = (\theta(t), \phi(t)), \ -\pi \le \theta(t) \le \pi, \ 0 \le \phi(t) \le \pi.$$

Thus, Equation (2), is reduced to

$$nL^j_k(t) = (\theta(t), \frac{\pi}{2}).$$

As confirmed in Fig. 4, $\theta(t)$ is the same both for $nH(t)$ and $nL^j_k(t)$ and gives their orientations with respect to nF^j_k.



(a) in-type (b) out-type

Fig. 4. The Movements of $nH(t)$ and $nL^j_k(t)$: (a) in-type (b) out-type

Remember that $nH(t)$ is a circular arc (or a circle) on the Gauss sphere that is generated by rotating $nH(0)$ counter-clockwisely about $-r$ by $\|r\|$ (or equivalently, clockwisely about r by $\|r\|$). Let

$$\tau^j_k = \cos^{-1}\langle \frac{-r}{\|r\|}, nH(t^j_e) \rangle \quad \text{and}$$

$$\delta^j_k = \cos^{-1}\langle \frac{-r}{\|r\|}, nF^j_k \rangle.$$

Depending on the relative positions of $nH(t^j_e)$ and nF^j_k with respect to $-r$, $\theta(t)$, $t^j_e \le t < t^j_l$, can be characterized as one of the following two types: $\theta(t)$, $t^j_e \le t < t^j_l$, is said to be of in-type if $\tau^j_k \ge \delta^j_k$; otherwise, it is of out-type (see Fig. 4). Clearly, the type classification can be done in constant time for each F^j_k. If $\theta(t)$, $t^j_e \le t < t^j_l$, is of in-type, then $nH(t)$ winds around nF^j_k. Therefore, $\theta(t)$ is monotone during the time interval $[t^j_e, t^j_l)$. Otherwise, $\theta(t)$ is not necessarily monotone. We later show that $\theta(t)$ in this case can be decomposed into at most three monotone curve segments in constant time.

Suppose that $\theta(t)$ is monotone for $[t_\alpha, t_\beta) \subset [t^j_e, t^j_l)$. From Theorem 2, we can find two extreme vertices v^j_α and v^j_β at t_α and t_β, respectively, in $O(\log m^j_k)$ time since F^j_k is given. If $\theta(t)$ is monotonically increasing, then $L^j_k(t)$ moves around F^j_k in the counter-clockwise order. Therefore, the vertices of F^j_k from v^j_α to v^j_β along the boundary in the same order gives the subsequence of extreme vertices during $[t_\alpha, t_\beta)$. Otherwise, the subsequence is those in the clockwise sense. In order to further improve efficiency, we implicitly give a 4-tuple $(v^j_\alpha, v^j_\beta, F^j_k, d)$ instead of explicitly listing all vertices between v^j_α and v^j_β. Here, d denotes the direction: either clockwise or counter-clockwise.

Now, we show how to check whether the monotone curve segment $\theta(t)$, $t_\alpha \le t < t_\beta$, is increasing or decreasing. This can be done by observing that the movement of $L^j_k(t)$ with respect to F^j_k. Notice that $nF^j_k = (0,0,1)$, i. e., it points to positive z-

axis. Let $nL_k^{j\prime}(t_\alpha)$ be the tangent vector of $nL_k^j(t)$ at t_α. If the z-component of the cross product, $nL_k^j(t) \times nL_k^{j\prime}(t)$, is positive, then $\theta(t)$ is increasing. Otherwise, $\theta(t)$ is decreasing. Obviously, this check can be done in $O(1)$ time. Hence, the following result holds true:

Lemma 5 If $\theta(t)$ is monotone for some time interval $[t_\alpha, t_\beta) \subset [t_e^j, t_l^j)$, then the subsequence of extreme vertices during $[t_\alpha, t_\beta)$ with respect to $H(t)$ can be found in $O(\log m_k^j)$ time, where m_k^j is the number of edges in $sreg(nF_k^j)$.

By Lemma 5, we can report, in $O(\log m_k^j)$ time, EV_j that is the subsequence of extreme vertices from F_k^j for $[t_e^j, t_l^j)$, if $\theta(t)$, $t_e^j \le t < t_l^j$ is of in-type. We show that the same time bound is achieved even if $\theta(t)$ is of out-type. Our approach is to decompose $\theta(t)$ at most three monotone curve segments in $O(1)$ time. We make use of $nH(t)$ to do it. $nH(t)$ rotates about $-r$ in the counterclockwise order to generate a circular arc on the Gauss sphere. Consider the orthogonal projection of $nH(t)$, $t_e^j \le t < t_l^j$, onto the xy-plane (see Fig. 4 (b)). Since $nH(t)$ is a circular arc, its projection is an elliptic arc. This projection preserves the component $\theta(t)$ of $nH(t)$. The projection of nF_k^j onto the plane is coincident with the origin of the plane. Since $\theta(t)$, $t_e^j \le t < t_l^j$, is of out-type, the origin is an exterior point of the ellipse that contains the elliptic arc $\overline{nH}(t)$ projected from $nH(t)$, $t_e^j \le t < t_l^j$. Therefore, there are two points at each of which a line passing the origin supports $\overline{nH}(t)$. Let $\overline{nH}(t_\alpha)$ and $\overline{nH}(t_\beta)$ be such points for $t_e^j \le t_\alpha \le t_\beta \le t_l^j$. If $[t_e^j, t_l^j)$ degenerates to a point, then $t_e^j = t_l^j$, and thus $t_\alpha = t_\beta$. In this case, EV_j consists of a single vertex that can be found in $O(\log m_k^j)$ time. Without loss of generality, we assume that $t_e^j \ne t_l^j$ to guarantee $t_\alpha \ne t_\beta$. There are four cases:

(1) $t_e^j = t_\alpha$ and $t_\beta = t_l^j$,

(2) $t_e^j = t_\alpha$ and $t_\beta < t_l^j$,

(3) $t_e^j < t_\alpha$ and $t_\beta = t_l^j$,

(4) $t_e^j < t_\alpha$ and $t_\beta < t_l^j$.

In case (1), $\theta(t)$ for $[t_e^j, t_l^j)$ is monotone in t. $\theta(t)$ is divided into two monotone curve segments: one over $[t_e^j, t_\beta)$ and the other over $[t_\beta, t_l^j)$ in case (2). Symmetrically, case (3) gives two monotone curve segments divided at t_α. Finally, in case (4), $\theta(t)$ is divided into three monotone curve segments defined over $[t_e^j, t_\alpha)$, $[t_\alpha, t_\beta)$, and $[t_\beta, t_l^j)$, respectively. Since $\theta(t)$, $t_e^j \le t < t_l^j$ consists of $O(1)$ monotone curve segments in every case, it takes $O(\log m_k^j)$ time to obtain EV_j regardless of the type of $\theta(t)$ due to Lemma 5.

Lemma 6 The subsequence EV_j of extreme vertices for time interval $[t_e^j, t_l^j)$ can be constructed in $O(\log m_k^j)$ time.

Now, we are ready to give how to solve our generalized extreme vertex problem in $O(\log n + \sum_{j=1}^{s} m_k^j)$ time. By Lemma 4, we obtain in $O(\log n + \sum_{j=1}^{s} m_k^j)$ time $T = ([t_e^0, t_l^0), [t_e^1, t_l^1), \cdots, [t_e^s, t_l^s))$. For each interval $[t_e^j, t_l^j)$, EV_j can be constructed in $O(\log m_k^j)$ time. Therefore, it takes $O(\sum_{j=0}^{s} \log m_k^j)$ time to find $EV = (EV_1, \cdots, EV_s)$. Hence, the result follows immediately.

Theorem 3 The sequence EV of all extreme vertices of a moving regular polyhedron P with respect to H can be found in $O(\log n + \sum_{j=1}^{s} m_k^j)$ time.

5. Concluding Remarks

Voronoi diagrams play a central role for solving a variety of proximity problems. We solve the extreme vertex problem of a translating polyhedron with respect to a fixed plane by transforming the problem to a point-location problem. It is accomplished by constructing the spherical

Voronoi diagram of the outward unit normal vectors of faces of the polyhedron in linear time. We extend this approach to efficiently solve the generalized version of the extreme vertex problem, where the polyhedron rotates about a fixed axis. We finally pose a new interesting problem: Can we solve the generalized problem in an output-sensitive way?

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김 형 석

1990년 연세대학교 이과대학 수학과(학사)
1992년 한국과학기술원 수학과 (석사, 매듭이론 전공)
1998년 한국과학기술원 수학과 (박사, 컴퓨터그래픽스 전공)
1998년~1999년 한국전자통신연구원 박사후연수연구원
1999년~현재 동의대학교 컴퓨터응용공학부 전임강사
관심분야 : 컴퓨터그래픽스, 계산기하학, 기하학적 모델링